Patterson-Sullivan Measures on the Circle

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Hyperbolic Space

Hyperbolic geometry differs from Euclidean geometry in the way that we measure distances between points. Namely, this affects the geodesics (shortest paths) between points and the notion of "straight lines." Using a conformal mapping, we can depict the entire complex plane as lying in the unit disk, where the "ruler" of distance shrinks as we move towards the boundary. Specifically, the a point r units away from the origin (to our eyes) really has distance

$$d(r,0)=\ln\left(\frac{1+r}{1-r}\right)=2\operatorname{arctanh}(r)$$

This realization of hyperbolic space is called the Poincaré disk [2]; we are interested in the properties of its boundary.

The Patterson-Sullivan Construction [3]

Given a group Γ acting on a point x in the hyperbolic plane, we define its orbit under Γ as

$$\Gamma\cdot x=\{\gamma\cdot x:\gamma\in\Gamma\}.$$

An example of an orbit on the point $0+i$ is pictured below.
Our goal is to characterize the proportion of the orbit that lies on the boundary of the disk. To do this, we first define the Poincaré series

$$Q_{\Gamma,x}(s)=\sum_{\gamma\in\Gamma}e^{-s\cdot d(x,\gamma\cdot x)}.$$

Here, d measures the hyperbolic distance between the basepoint x and the orbit point $\gamma\cdot x$. We then define

$$\delta=\inf\{s\geq 0:Q_{\Gamma,x}<\infty\}.$$

Intuitively, δ is the "dividing line" of convergence, and so it is aptly named the **critical exponent** of Γ. We remark that δ is the Hausdorff dimension of the boundary as well, and that the series need not converge at δ. Finally, we can define a sequence of measures

$$\mu_{x,s}=\frac{1}{Q_{\Gamma,x}}\sum_{\gamma\in\Gamma}e^{-s\cdot d(x,\cdot x)}D_{\gamma\cdot x},$$

where $D_{\gamma\cdot x}$ is an indicator of the set. This results in adding only the orbit points belonging to the set, and then normalizing by the total computed previously. Finally, we can take the associated limit

$$\mu_x=\lim_{s\to\delta^+}\mu_{x,s}$$

which yields the measure we are after. In the remainder of this poster, we explore examples of the measure and describe two different methods of calculating such measures numerically.

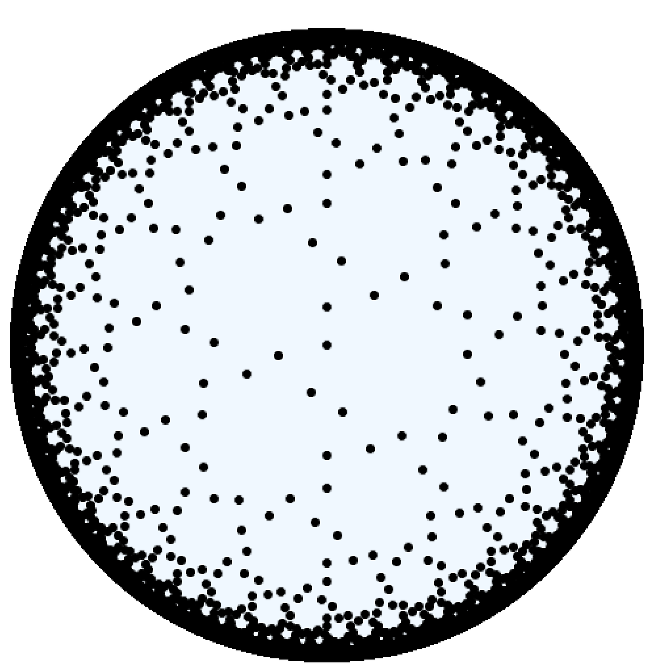

Fig. 1: The orbit of $0+i$ under the Coxeter triangle group (2,3,7)

Method

As a first approach, we used the geometry_tools Python library [4] to efficiently generate truncated orbits under free groups and Coxeter groups. Because these groups have infinite elements, we only generate elements which are products of up to 25 generators; finite-state automata can be utilized to efficiently enumerate such elements. With the orbits in hand,

the calculation proceeds as in the previous section. However, since we only have finitely many group elements, we cannot test for convergence directly. Luckily, we can approximate δ by fitting an exponential curve to the histogram of distances.

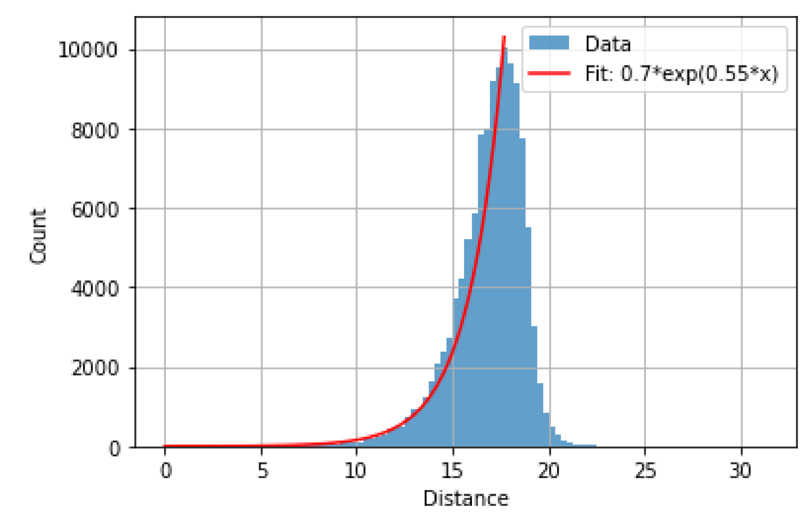

Fig. 2: A histogram of distances from orbit points to the basepoint $0+1i$.

Once we have δ, we select a tolerance for how close points need to be to the edge to be included, and from here we can calculate the measure over our sample.

The McMullen Algorithm

An algorithm for computing δ is outlined in a paper [1] of McMullen. The general idea is that we partition the boundary into subsets, each with a designated function. We then compute the preimages of these sample points under the generators and use these to refine our partition further. With each step we can also compute an approximation to δ which converges exponentially with the number of iterations.

We start with a partition $\langle(P_i,f_i)\rangle$ of the set under consideration, where each f_i is a map defined on P_i. Such an object is called a **Markov Partition**. Each P_i is also equipped with an associated sample point x_i. We say that $i\mapsto j$ iff $f(P_i)\cap P_j\neq\emptyset$. We then repeat these steps ad libitum:

1. If $i\mapsto j$, find the $y_{ij}\in P_i$ such that $f_i(y_{ij})=x_j$.
2. Construct the matrix given by
$$T_{ij}=\begin{cases}|f_i'(y_{ij})|^{-1} & :i\mapsto j\\ 0 & :i\not\mapsto j\end{cases}$$
3. Find the real α such that the largest eigenvalue-magnitude of $T^\alpha=1$, where the power is taken entrywise.
4. Construct the refinement $\langle(R_{ij},f_i)\rangle$ where
$$R_{ij}=f_i^{-1}(P_j)\cap P_i.$$

In practice, our matrix free group affects the mobius transformation

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}z=\frac{az+b}{cz+d}$$

with associated derivative

$$f'(z)=\frac{ad-bc}{(cz+d)^2}=\frac{1}{(cz+d)^2}$$

since our generators have determinant 1. Having invertible generators also makes it much easier to compute pre-images in this special case. We also note that one can find α numerically using a standard root-finding algorithm found in libraries like SciPy.

Results

Using the first method, we have successfully computed the critical exponent δ of the free group on the matrix generators

$$A=\begin{pmatrix}3 & 0\\ 0 & \frac{1}{3}\end{pmatrix}\qquad B=\begin{pmatrix}\frac{5}{3} & \frac{4}{3}\\ \frac{4}{3} & \frac{5}{3}\end{pmatrix}$$

with a value of $\delta=0.55$. We also recovered the value $\delta=1$ on the general Coxeter triangle group. As for the boundary measure, we have the estimates $\mu=0.1$ and $\mu=1$ for the respective groups, but are less confident in the validity of these results. As the first method requires choosing a tolerance (our points can never reach the boundary on a computer), the proportion of accumulation points is difficult to measure directly.

Further Steps

While we did perform some Monte-Carlo sampling during initial investigation of the problem, we never implemented it into the final pipeline for computing μ using the first method. This could possibly give us a more accurate estimation as it allows for enumeration of group elements with a much higher maximum length.
Unfortunately, we have yet to finish the full implementation of the McMullen algorithm, so completing this would be a top priority in future work. We remark that μ can be computed in the process of running this algorithm, and so an implementation would yield a potentially more accurate method of calculating the measure.
Another avenue for progress would be to extend this work to different classes of groups acting in hyperbolic space.

Acknowledgements

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References

[1] Curtis T. McMullen. *Hausdorff dimension and conformal dynamics III: Computation of dimension*. 1997. URL: https://people.math.harvard.edu/~ctm/papers/home/text/papers/dimIII/dimIII.pdf.

[2] *Poincaré disk model*. URL: https://en.wikipedia.org/wiki/Poincar%C3%A9_disk_model. (accessed: 12.5.2024).

[3] J.F. Quint. *An overview of Patterson-Sullivan Theory*. URL: https://www.math.u-bordeaux.fr/~jquint/publications/courszurich.pdf.

[4] Teddy Weisman. *geometry_tools package*. URL: https://github.com/tjweisman/geometry_tools.