Mutations in Number Theory

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The Markov Equation

Consider the Diophantine **Markov Equation**:

$$x_1^2 + x_2^2 + x_3^2 = 3x_1x_2x_3$$

- What do the (integer-valued) solutions look like?
- Can you find an easy one?
- Is there an efficient way to enumerate them?

Mutation of Variables

Starting from a solution (x_1, x_2, x_3), we can get a new solution by **mutating** in any of the components:

$$x_1' = \frac{x_2^2 + x_3^2}{x_1}$$

$$x_2' = \frac{x_3^2 + x_1^2}{x_2}$$

$$x_3' = \frac{x_1^2 + x_2^2}{x_3}$$

For instance,

$$(1, 5, 2) \xrightarrow{\mu_3} (1, 5, 13)$$

Nice Properties of Mutations

- Mutations have the following property:

$$\frac{x_1^2 + x_2^2 + x_3^2}{x_1 x_2 x_3} = \frac{(x_1')^2 + x_2^2 + x_3^2}{x_1' x_2 x_3}$$

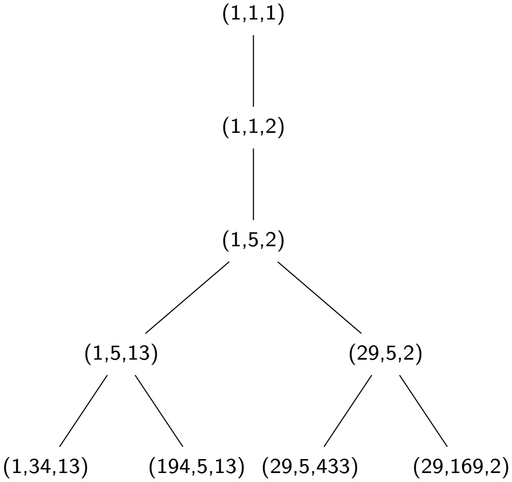
  Mutations indeed send solutions to solutions!

- Mutation is an involution:

$$(x_1')' = \frac{x_2^2 + x_3^2}{x_1'} = \frac{(x_2^2 + x_3^2)x_1}{x_2^2 + x_3^2} = x_1$$

Tree of Triples

If we start with the triple $(1, 1, 1)$ and apply mutations in all directions, obtain a tree of solution triples:



Every solution to the Markov equation lies on this tree!

Towards Cluster Mutations

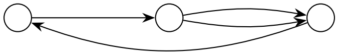
The previous methods are reminiscent of a general pattern:

- Start with some distinguished tuple of values
- Mutate values according to relations in terms of the other values
- Use the behavior of the mutations to say something about the tuples

We introduce **Cluster Mutations**, a way to organize these mutations in a combinatorial manner.

Quivers

The directed graph below is called a **Quiver** because it has no arrows going in opposite directions (note that multiple arrows between vertices are allowed)



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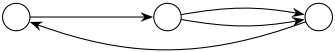
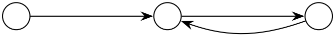


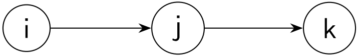
The below graph is **not** a quiver (why?):



Mutations

Given a quiver, we can **mutate** at a vertex j by doing the following:

1. For every distinct path $i \to j \to k$, add an arrow $i \to k$
2. Flip all arrows incident to j
3. Remove any resulting 2-cycles



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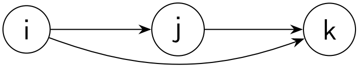

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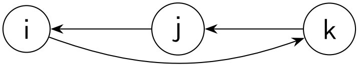

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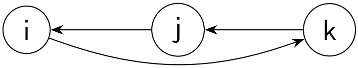
Quiver mutation is an involution:



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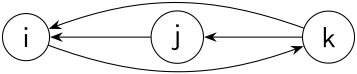

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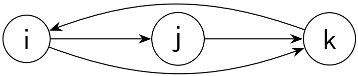

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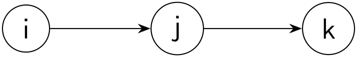

This is our original quiver!

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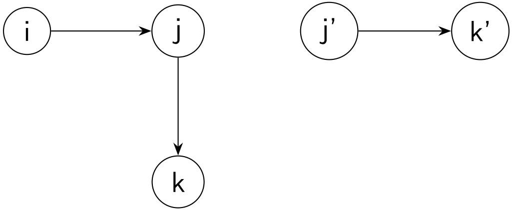
Mutations at non-adjacent vertices commute:



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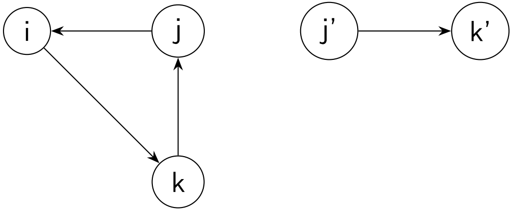

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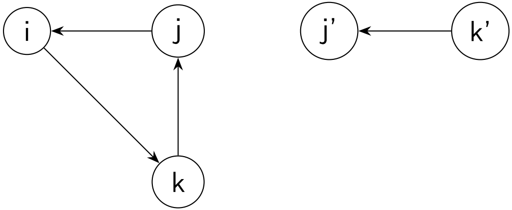
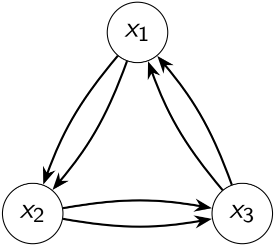


Clusters and Seeds

- We can assign variables $x_1, x_2, \ldots$ to the vertices of a quiver
- The quiver below is called the **Markov Quiver** (why?)

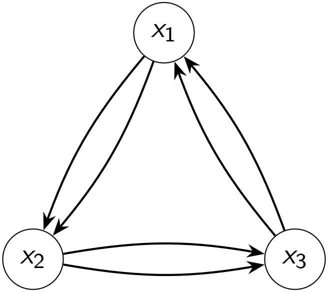


- We call the tuple of variables $(x_1, \ldots, x_n)$ the **cluster**
- The cluster and quiver together are called a **seed**

Mutation of Seeds

To mutate the seed at x_j:

- Mutate the quiver at x_j



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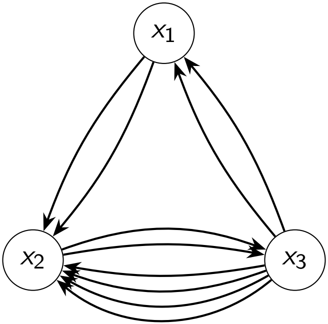

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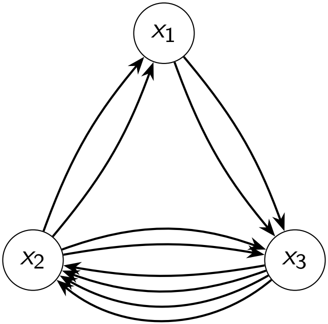

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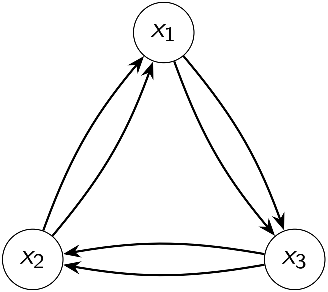
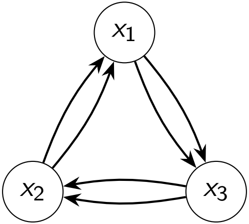


Mutation of Seeds

To mutate the seed at x_j:

- Mutate the quiver at x_j
- Mutate the variable x_j according to the formula:

$$x_j' x_j = \underbrace{\prod_{b_{ij}>0} x_i^{b_{ij}}}_{\text{arrows into } x_j} + \underbrace{\prod_{b_{ji}>0} x_i^{b_{ji}}}_{\text{arrows out of } x_j}$$



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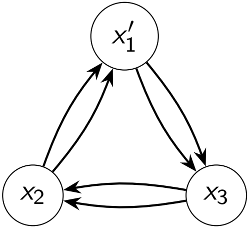
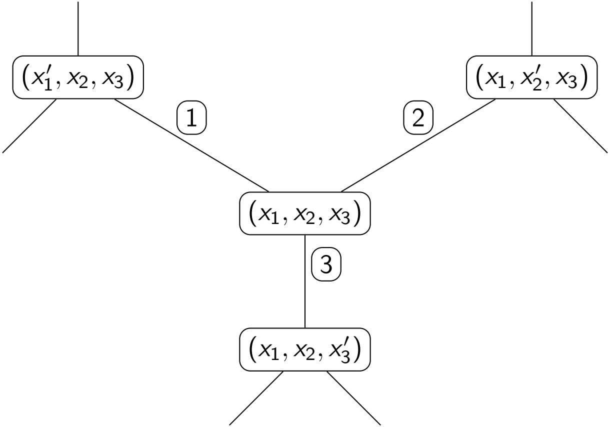


$$x_1' = \frac{x_2^2 + x_3^2}{x_1}$$

This is exactly the variable mutations which generate the Markov triples!

Tree of Mutations

From the initial seed with cluster $(x_1, \ldots, x_n)$, we again have a tree of possible mutation sequences:



Putting it all Together

We can now cast our original example in the language of clusters:

- Start with the seed consisting of the Markov quiver and the cluster (x_1, x_2, x_3)
- Take all possible mutation sequences over x_1, x_2, x_3
- Specialize at $x_1 = x_2 = x_3 = 1$ to set the invariant correctly; the clusters now give solutions to the Markov equation

The Somos-4 Sequence

The **Somos-4 Sequence** is a recurrence defined by

$$x_{m+2}x_{m-2} = x_{m+1}x_{m-1} + x_m^2$$

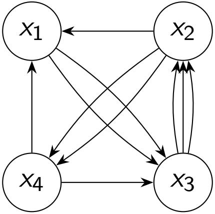
with initial conditions $x_1 = x_2 = x_3 = x_4 = 1$.
The terms of this sequence look like

$$1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, 8209, 83313, 620297, \ldots$$

Are all of them integers?

The Somos Quiver

- We would like a seed that generates this sequence
- Consider the initial cluster (x_1, x_2, x_3, x_4) and the quiver below:



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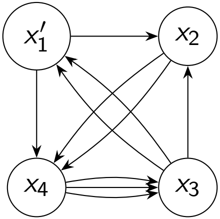
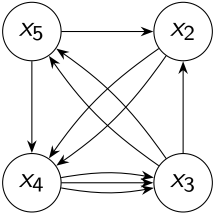


- Mutation at x_1 is just a rotation
- Mutating at x_2 now would rotate again...

Recovering the Recurrence



$$x_5 := x_1' \qquad x_5 = \frac{x_2 x_4 + x_3^2}{x_1}$$

By symmetry, the mutation sequence $x_1, x_2, x_3, x_4, x_5, x_6 \ldots$ always gives the same formula:

$$x_{m+2} x_{m-2} = x_{m+1} x_{m-1} + x_m^2$$

So this path through the mutation tree generates the Somos-4 sequence!

Applying the Laurent Phenomenon

Theorem

For any fixed cluster $(x_1, \ldots, x_n)$, any variable in any cluster can be expressed as a Laurent polynomial with integer coefficients in variables $x_1, \ldots, x_n$.

- Every cluster variable is a Laurent polynomial with integer coefficients in the variables x_1, x_2, x_3, x_4.
- Specialize $(x_1, x_2, x_3, x_4) = (1, 1, 1, 1)$ to agree with the initial conditions of the recurrence
- A Laurent polynomial in 1 with integer coefficients is just an integer!

Theorem

Every element of the Somos-4 sequence is an integer.

Takeaways

- "Mutation operations" which appear throughout mathematics can be described in a combinatorial way with Quivers
- Clusters add an algebraic structure to Quivers, allowing one to prove algebraic facts using mutations
- Working backwards to find a generating seed is a technique to study certain sequences

Thank you!

Questions?